

Quantum Supergroups And Topological Invariants of Three - Manifolds

R. B. Zhang

Department of Pure Mathematics
University of Adelaide
Adelaide, SA 5001, Australia

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The Reshetikhin - Turaeve approach to topological invariants of three - manifolds is generalized to quantum supergroups. A general method for constructing three - manifold invariants is developed, which requires only the study of the eigenvalues of certain central elements of the quantum supergroup in irreducible representations. To illustrate how the method works, $U_q(gl(2|1))$ at odd roots of unity is studied in detail, and the corresponding topological invariants are obtained.

1 Introduction

The discovery of the Jones invariants[1] marked the beginning of a new phase in knot theory. Soon after Jones discovery, it was realized[2] that these invariants are intimately connected with soluble models in statistical mechanics[3] through the Yang - Baxter equation[4][3]. Subsequent investigations on this connection[5][6] turned out to be very fruitful. The connection of knot theory and physics went much further. In a seminal paper, Witten interpreted the Jones and related link polynomials in terms of the quantum Chern - Simons theory[7]. The virtue of Witten's formulation is that it is intrinsically three dimensional, thus topological invariants of three manifolds are obtained in the same token.

In the study of knot theory and the Yang - Baxter equation, quantum groups[8][9] and quantum supergroups[10] played an important role. A quantum (super)group $U_q(g)$ has the properties of a (\mathbf{Z}_2 graded) quasi triangular Hopf algebra, admitting a universal R - matrix, which satisfies the Yang - Baxter equation. Categorical arguments show[11] that every tangle can be associated with a homomorphism of finite dimensional representations of $U_q(g)$, in a manner consistent with the isotopy properties of the former. In particular, a link is associated with a map from the trivial $U_q(g)$ module to itself, which yields a regular isotopy invariant of the link(a simple scaling of the map produces an ambient isotopy invariant)[12]. A more down to earth approach[13] to link polynomials, which is closer to Jones' original construction in spirit, uses the fact that in any finite dimensional irreducible representation of $U_q(g)$, the R - matrix generates a representation of the braid group, on which a Markov trace can be defined in a general fashion. In this way, each finite dimensional irreducible representation of any quantum (super)group leads to a link invariant, and all the well known link polynomials can be obtained as special cases. For example, the Homfly polynomial[14] arises from the vector representations of $U_q(gl(m))$ and $U_q(gl(m|n))$, $m \neq n$ (with the Jones polynomial as a special case), the Alexander - Conway polynomial from the vector representation of $U_q(sl(m|m))$, and the Kauffman polynomial from the vector representations of $U_q(so(n))$ and $U_q(osp(M|2n))$.

It is well known that the study three - manifold topology may be reduced to the study of links embedded in S^3 . In the early 1960s, Lickorish and Wallace established a theorem stating that each framed link in S^3 determines a compact, closed, oriented 3 - manifold, and every such 3 - manifold is obtainable by surgery along a framed link in S^3 [15]. The problem with this description of 3 - manifolds is that it is not unique: different framed links may yield homeomorphic 3 - manifolds upon surgery. This problem was resolved by Kirby[16] and Crags, whose result was further refined by Fenn and Rourke in [17]. These authors proved that orientation preserving homeomorphism classes of compact, closed, oriented 3 - manifolds correspond bijectively to equivalence classes of framed links in S^3 , where the equivalence relation is generated by the Kirby moves.

In view of these results, one naturally expects it to be possible to construct topological invariants of 3 - manifolds using the link polynomials arising from quantum (super)groups as building blocks, and this is indeed the case. In a seminal paper[18], Reshetikhin and Turaev developed a purely algebraic approach to three - manifold invariants based on the theory of quantum groups. The essential idea of [18] is to make appropriate combinations of isotopy invariants of a framed link embedded in S^3 , such

that they will be intact under the Kirby moves, thus qualify as topological invariants of the three - manifold obtained by surgery along this link. In [18], a method was also devised for explicitly constructing such combinations. Applying it to the quantum group $U_q(sl(2))$ at even roots of unity led to a set of three - manifold invariants, which turned out to be equivalent to those obtained by Witten using $sl(2)$ Chern - Simons theory[19] – [22]. The application of this method in general requires detailed analysis of indecomposable representations of, typically, a quantum group at roots of unity, a problem of a very high degree of difficulty. However, guided by properties of Witten's Chern - Simons theory, Turaev and Wenzl in a remarkable paper[24] resolved the problem for $U_q(gl(m))$ and $U_q(so(n))$ at roots of unity, thus proved the existence of the three - manifold invariants associated to these quantum groups.

The algebraic approach of Reshetikhin and Turaev was generalized to the quantum supergroup $U_q(osp(1|2))$ at odd roots of unity in [23], and the corresponding three - manifold invariants were constructed. In that paper, a way was found to avoid the difficult problem of classifying indecomposable representations. The construction there required only the study of the eigenvalues of certain central elements of the algebra in the irreducible representations.

We should point out that the Witten - Reshetikhin - Turaev invariant was also provided with an elementary derivation by Lickorish[25], who utilized only properties of the Temperley - Lieb algebra. We should also mention that there exists another well known 3 - manifold invariant due to Tureav and Viro[26], which was obtained from the quantum group analog of Ponzano and Regge's formula for the partition function of three - dimensional quantum gravity on a lattice[27]. The various approaches to 3 - manifold invariants are undoubtedly related in an intimate way, and a coherent understanding of their connection will be very interesting.

The aim of this paper is to generalize the Reshetikhin - Turaev approach to \mathbf{Z}_2 graded Hopf algebras, thus to erect a general framework for constructing three - manifold invariants using quantum supergroups. As is well known, the representation theory of quantum supergroups is drastically different from that of their nongraded counter parts, hence we expect that quantum supergroups will yield three - manifold invariants genuinely different from those derived from ordinary quantum groups. Another fact worth noticing is that there exist severe difficulties in developing a proper formulation of Chern - Simons theory with super gauge groups[28]. Therefore the algebraic approach seems to be a more practicable way to the construction of three - manifold invariants using supergroups.

A method for constructing Kirby move invariant combinations of link polynomials will be given in section 4, which, as we will see, essentially boils down to finding a set of constants d_λ 's, which render vanishing the central element δ (see equation (14)) of the employed \mathbf{Z}_2 graded Hopf algebra A in all irreducible A modules with nonzero q - superdimensions. This construction is applied to the quantum supergroup $U_q(gl(2|1))$ at odd roots of unity in section 5, and the corresponding three - manifold invariants are obtained. To make our paper self contained, we review the basic properties of coloured ribbon graphs[11], and also present some notions of \mathbf{Z}_2 - graded Hopf algebras(i.e., Hopf superalgebras).

2 \mathbf{Z}_2 graded Hopf algebras

Let us begin by quickly reviewing the definition of a \mathbf{Z}_2 graded Hopf algebra. We will work in the complex field \mathbf{C} throughout the paper. A \mathbf{Z}_2 graded vector space over \mathbf{C} is a direct sum of two subspaces $V = V_0 \oplus V_1$, with V_0 and V_1 called the even and odd subspaces respectively. We introduce the gradation index $[] : V_0 \cup V_1 \rightarrow \mathbf{Z}_2 = \{0, 1\}$, such that if $x \in V_i$, then $[x] = i$. We will call an element of V homogeneous if it belongs to $V_0 \cup V_1$, and inhomogeneous otherwise. The dual space V^* of V is also \mathbf{Z}_2 graded, $V^* = V_0^* \oplus V_1^*$, where $V_i^* = \text{Hom}_{\mathbf{C}}(V_i, \mathbf{C})$, $i \in \mathbf{Z}_2$. The tensor product $V \otimes_{\mathbf{C}} W$ of two \mathbf{Z}_2 graded vector spaces V and W inherits a \mathbf{Z}_2 grading from that of V and W , with $V \otimes W = \bigoplus_{i \in \mathbf{Z}_2} (V \otimes W)_i$, $(V \otimes W)_i = \bigoplus_k V_k \otimes W_{i-k(\text{mod}2)}$.

A linear map $f : V \rightarrow W$ of two \mathbf{Z}_2 graded vector spaces V and W is said to be homogeneous of degree $r \in \mathbf{Z}_2$ if $f(V_i) \subseteq W_{i+r(\text{mod}2)}$. We will call f a homomorphism if $r = 0$, an isomorphism if it is also one - to - one and onto. Note that the linear homogeneous map $f : V \rightarrow W$ is uniquely defined by specifying the images of the homogeneous elements of V , then extending the definition to the entire space through linearity. For later use, we now define the twisting homomorphism

$$T : V \otimes W \rightarrow W \otimes V. \quad (1)$$

All we need to do is to designate for any homogeneous elements $x \in V$ and $y \in W$ that

$$T(x \otimes y) = (-1)^{[x][y]} y \otimes x,$$

and then extends T to all elements of $V \otimes W$ through linearity.

A \mathbf{Z}_2 graded algebra A is a \mathbf{Z}_2 graded vector space equipped with homomorphisms $M : A \otimes A \rightarrow A$, and $e : \mathbf{C} \rightarrow A$, respectively called the multiplication and unit, such that M is associative, i.e., $M(M \otimes id) = M(id \otimes M)$, and $e(c)a = ae(c) = ca$, where $c \in \mathbf{C}$, $a \in A$, and $id : A \rightarrow A$ is the identity map. Note that the tensor product $A \otimes_{\mathbf{C}} B$ of two \mathbf{Z}_2 graded algebras is again a \mathbf{Z}_2 graded algebra, with the multiplication given by

$$(a_1 \otimes b_1).(a_2 \otimes b_2) = (-1)^{[a_2][b_1]} a_1 a_2 \otimes b_1 b_2,$$

where $a_1, a_2 \in A$, $b_1, b_2 \in B$. Let $f : A \rightarrow B$ be a homomorphism of the underlying \mathbf{Z}_2 graded vector spaces. We call f a \mathbf{Z}_2 graded algebra homomorphism if for $a_1, a_2 \in A$,

$$f(a_1 a_2) = f(a_1) f(a_2),$$

and an anti - homomorphism if

$$f(a_1 a_2) = (-1)^{[a_1][a_2]} f(a_2) f(a_1).$$

A \mathbf{Z}_2 graded co - algebra B is a \mathbf{Z}_2 graded vector space equipped with homomorphisms $\Delta : B \rightarrow B \otimes_{\mathbf{C}} B$, and $\epsilon : B \rightarrow \mathbf{C}$, respectively called the co - multiplication and co - unit, such that Δ is co - associative, i.e.,

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta,$$

and ϵ has the following unitarity property

$$(\epsilon \otimes id)\Delta = (id \otimes \epsilon)\Delta = id.$$

The tensor product $B \otimes_{\mathbf{C}} A$ of two \mathbf{Z}_2 graded co - algebras B and A is again a \mathbf{Z}_2 graded co - algebra with the co - multiplication $(id_B \otimes T \otimes id_A)(\Delta_B \otimes \Delta_A)$ and co - unit $\epsilon_B \otimes \epsilon_A$.

Let A be a \mathbf{Z}_2 graded algebra with multiplication M and unit e , and at the same time also be a \mathbf{Z}_2 graded co - algebra with co - multiplication Δ and co - unit ϵ . We call A a \mathbf{Z}_2 graded bi - algebra if both Δ and ϵ are \mathbf{Z}_2 graded algebra homomorphisms. If further, there exists an anti - homomorphism $S : A \rightarrow A$, such that $M(id \otimes S)\Delta = M(S \otimes id)\Delta = \epsilon$, then A is called a \mathbf{Z}_2 graded Hopf algebra, or a Hopf superalgebra.

Let us introduce the category of finite dimensional linear representations Rep_A of A , of which the objects are the finite dimensional left A modules over \mathbf{C} , and the morphisms are A linear homomorphisms. Note that a given object V in Rep_A is a \mathbf{Z}_2 graded vector space, i.e., $V = V_0 \oplus V_1$. Let a be a homogeneous element of A with $[a] = i$, then

$$a : V_j \rightarrow V_{i+j(mod2)}.$$

Given two objects V and W of Rep_A , the \mathbf{Z}_2 graded vector space $V \otimes_{\mathbf{C}} W$ also belongs to Rep_A , with the action of $a \in A$ defined by

$$a(x \otimes y) = \sum_{(a)} (-1)^{[a_{(2)}][x]} a_{(1)}x \otimes a_{(2)}y, \quad x \in V, y \in W,$$

where Sweedler's sigma notation

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}, \quad a \in A,$$

has been employed. For any A module V , the dual vector space V^* is also an A module with the action of $a \in A$ defined by

$$(ax^*)(y) = (-1)^{[a][x^*]} x^*(S(a)y), \quad x \in V^*, \quad y \in V.$$

By the trivial A module we shall mean the one dimensional module $\mathbf{C}x$ such that $ax = \epsilon(a)x, \forall a \in A$.

A quasi triangular \mathbf{Z}_2 graded Hopf algebra (A, R) is a Hopf superalgebra A , which admits an even element $R \in A \otimes A$, called the universal R matrix, satisfying the following relations

$$R\Delta(a) = \Delta'(a)R, \quad \forall a \in A, \quad (2)$$

$$\begin{aligned} (\Delta \otimes id)R &= R_{13}R_{23}, \\ (id \otimes \Delta)R &= R_{13}R_{12}, \end{aligned} \quad (3)$$

where $\Delta' = T\Delta$. It is not difficult to show that the above equations imply that R obeys the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \quad (4)$$

and also satisfies

$$R^{-1} = (S \otimes id)R = (id \otimes S^{-1})R.$$

Express the universal R matrix as $R = \sum_t \alpha_t \otimes \beta_t$, and define

$$u = \sum_t (-1)^{[\alpha_t]} S(\beta_t) \alpha_t.$$

Then

$$\begin{aligned} u^{-1} &= \sum_t (-1)^{[\alpha_t]} \beta_t S^2(\alpha_t), \\ S^2(a) &= uau^{-1}, \quad \forall a \in A. \end{aligned} \tag{5}$$

A ribbon Hopf superalgebra $(A, R, K_{2\rho})$ is a quasi triangular Hopf superalgebra (A, R) with a group like element $K_{2\rho}$ (i.e., invertible and $\Delta(K_{2\rho}) = K_{2\rho} \otimes K_{2\rho}$), such that

$$S^2(a) = K_{2\rho} a K_{2\rho}^{-1}, \quad \forall a \in A.$$

Define

$$v = u K_{2\rho}^{-1}. \tag{6}$$

It can be proved that v lies in the central algebra of A , and

$$\Delta(v) = (v \otimes v)(R^T R)^{-1},$$

where $R^T = T(R)$.

Let V be a finite dimensional A module. We denote the corresponding representation by π . Define[29]

$$C_V = \text{Str}_V[(\pi \otimes id)(K_{2\rho}^{-1} \otimes 1)R^T R], \tag{7}$$

where Str_V represents the supertrace taken over V . Then C_V belongs to the central algebra of A . The eigenvalue of C_V in the trivial irrep of A is called the q - superdimension of V , which we denote by $SD_q(V)$.

3 Coloured ribbon graphs

3.1 Coloured ribbon graphs

Coloured ribbon graphs were discussed extensively by Reshetikhin and Turaev in [11] and [18]. In order to make the present paper self contained, we rephrase some of the basic notions here, and refer to the above mentioned papers for further details.

By a ribbon we mean the square $[0, 1] \times [0, 1]$ smoothly embedded in R^3 . The images of $[0, 1] \times 0$ and $[0, 1] \times 1$ are the bases, and that of $\frac{1}{2} \times [0, 1]$ is called the core of the ribbon. Similarly an annulus is the cylinder $S^1 \times [0, 1]$ embedded in R^3 , and the image of $S^1 \times \frac{1}{2}$ under the embedding is called the core of the annulus. Ribbons and annuli are oriented as surfaces and their cores are directed.

Given $k, l \in \mathbf{Z}_+$. A (k, l) ribbon graph is an oriented surface consisting of ribbons and annuli such that ribbons and annuli never meet, and this surface intersects $(\mathbf{R}^2 \times 0) \cup (\mathbf{R}^2 \times 1)$ in the bases of the ribbons, where the collection of these bases are the

collection of segments $\{[i - \frac{1}{4}, i + \frac{1}{4}] \times 0 \times 0 | i = 1, 2, \dots, k\} \cup \{[j - \frac{1}{4}, j + \frac{1}{4}] \times 0 \times 1 | j = 1, 2, \dots, l\}$. For simplicity, we will represent a ribbon or an annulus by its directed core.

We introduce two operations, composition and juxtaposition, to manufacture new ribbon graphs from given ones. Given (k, l) graph Γ_1 , (l, m) graph Γ_2 , and (k', l') graph Γ_3 , the composition of $\Gamma_1 \circ \Gamma_2$ is defined in the following way: shift Γ_2 by the vector $(0, 0, 1)$ into $\mathbf{R}^2 \times [1, 2]$, glue the bottom end of Γ_2 to the top end of Γ_1 in such a way that the core of the ribbons glued together should have the same direction as the core of the resultant ribbon (if this is not possible, then the composition is not defined.), then reduce the size of the resultant picture by a factor of 2, leading to a (k, m) graph. The juxtaposition $\Gamma_1 \otimes \Gamma_3$ is to position Γ_3 on the right of Γ_1 , leading to a $(k + k', l + l')$ graph.

By repeatedly applying these two operations to the set of ribbon graphs depicted in Figure 1, we can generate all the ribbon graphs:

Figure 1.

We associate to each (k, l) graph Γ two sequences $\epsilon_*(\Gamma) = (\epsilon_1, \epsilon_2, \dots, \epsilon_k)$ and $\epsilon^*(\Gamma) = (\epsilon^1, \epsilon^2, \dots, \epsilon^l)$, $\epsilon^i, \epsilon_j \in \{1, -1\}$, in the following way. For a ribbon of Γ with a base $[i - \frac{1}{4}, i + \frac{1}{4}] \times 0 \times 0$ (resp. $[j - \frac{1}{4}, j + \frac{1}{4}] \times 0 \times 1$), if its core is directed towards (resp. away from) this base, then $\epsilon_i = 1$ (resp. $\epsilon^j = 1$), and $\epsilon_i = -1$ (resp. $\epsilon^j = -1$) otherwise.

Let $(A, R, K_{2\rho})$ be a ribbon Hopf superalgebra. Let $\{V_i | i \in I\}$ be a set of finite dimensional A modules. We also introduce the set \mathcal{N} consists of finite sequences of the form $((i_1, \epsilon_1), (i_2, \epsilon_2), \dots, (i_k, \epsilon_k))$, $k \in \mathbf{Z}_+$, $\epsilon_t \in \{1, -1\}$.

A colouring of a ribbon graph Γ is a mapping col associating with each ribbon or annulus of Γ an index $i \in I$. The category \mathcal{H} of coloured ribbon graphs is defined to have as objects the elements of \mathcal{N} , and as morphisms the coloured ribbon graphs, where we require that if the coloured ribbon graph Γ is a morphism $\eta \rightarrow \eta'$, $\eta, \eta' \in \mathcal{N}$, then the sequences of colours and directions of cores of the bottom and top ribbons must be equal to η and η' respectively. \mathcal{H} is provided with a tensor product structure $\otimes : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$, such that the tensor product of objects η and η' is to position the latter on the right of η to form one sequence, and the tensor product of morphism is simply the juxtaposition of ribbon graphs defined earlier.

3.2 The Reshetikhin - Turaev functor

Reshetikhin and Turaev[11] proved that there exists a unique covariant functor from \mathcal{H} to the category of finite dimensional representations of ribbon Hopf algebras, e.g., quantum groups. Their result can be directly generalized to ribbon Hopf superalgebras, leading to the following theorem

Theorem 1 *Let $(A, R, K_{2\rho})$ be a ribbon Hopf superalgebra, $\{V_i | i \in I\}$ a set of finite dimensional A modules, and \mathcal{H} the corresponding category of coloured ribbon graphs. Then there exists a unique covariant functor $F : \mathcal{H} \rightarrow Rep_A$ such that*

1. F transform any object $\eta = ((i_1, \epsilon_1), (i_2, \epsilon_2), \dots, (i_k, \epsilon_k))$ of \mathcal{H} into the A module $V(\eta) = V_{i_1}^{\epsilon_1} \otimes V_{i_2}^{\epsilon_2} \otimes \dots \otimes V_{i_k}^{\epsilon_k}$, where $V_i^{+1} = V_i$, $V_i^{-1} = V_i^*$, and if $k = 0$, then $V(\eta)$ is defined as the trivial A module.

2. For any two coloured ribbon graphs Γ and Γ' ,

$$F(\Gamma \otimes \Gamma') = F(\Gamma) \otimes F(\Gamma'), \quad (8)$$

that is, F preserves the tensor product operation.

3. Colour the bottom left ribbons of X^+ and X^- by i and the bottom right ones by j , and denote the resultant coloured ribbon graphs by X_{ij}^+ and X_{ij}^- respectively, then

$$\begin{aligned} F(X_{ij}^+) &= PR : V_i \otimes V_j \rightarrow V_j \otimes V_i, \\ F(X_{ij}^-) &= R^{-1}P : V_i \otimes V_j \rightarrow V_i \otimes V_j, \end{aligned} \quad (9)$$

where R is the universal R matrix of A , and P is the graded permutation operator. Similarly, we have

$$\begin{aligned} F(I_i^+) &= id : V_i \rightarrow V_i, \\ F(I_i^-) &= id : V_i^* \rightarrow V_i^*, \end{aligned} \quad (10)$$

and

$$\begin{aligned} F(\Omega_i^+) : V_i^* \otimes V_i &\rightarrow \mathbf{C}, \\ x^* \otimes y &\mapsto x^*(y), \\ F(\Omega_i^-) : V_i \otimes V_i^* &\rightarrow \mathbf{C}, \\ x \otimes y^* &\mapsto (-1)^{[x]} y^*(K_{2\rho} x), \\ F(U_i^+) : \mathbf{C} &\rightarrow V_i \otimes V_i^*, \\ c &\mapsto c \sum_t b_t \otimes b_t^*, \\ F(U_i^-) : \mathbf{C} &\rightarrow V_i^* \otimes V_i, \\ c &\mapsto c \sum_t (-1)^{[b_t]} b_t^* \otimes K_{2\rho}^{-1} b_t, \end{aligned} \quad (11)$$

where $\{b_t\}$ is a basis for V_i , and $\{b_t^*\}$ a basis for V_i^* , which are dual to each other in the sense that $b_s^*(b_t) = \delta_{st}$.

An important property of the functor F [11] is that if Γ is a coloured ribbon (k, k) graph, and $\hat{\Gamma}$ is the coloured ribbon $(0, 0)$ graph obtained by closing Γ , then

$$F(\hat{\Gamma}) = str[\Delta^{(k-1)}(K_{2\rho})F(\Gamma)],$$

where the supertrace is taken over the tensor product of the A modules colouring the open strands of Γ . In particular, if Γ is a $(1, 1)$ graph with the open strand coloured by such an A module V that all central elements of A act on it as scalars, then

$$\begin{aligned} F(\Gamma) &= \gamma id_V : V \rightarrow V, \\ F(\hat{\Gamma}) &= \gamma SD_q(V), \end{aligned}$$

where γ is a constant. It follows that for any $(0,0)$ graph, if any of its components is coloured with an A module (on which central elements of A act on as scalars,), which has a vanishing q - superdimension, then the Reshetikhin - Turaev functor yields zero when applied to it.

Let us now consider as examples the ribbon (k, k) graphs depicted in Figure 2.

Figure 2.

We colour the bottom ribbons of both Figure 2.a and 2.b by $((i_1, +1), (i_2, +1), \dots, (i_k, +1))$, while colour the annulus of Figure 2.a by such an element j of I that the associated A module V_j has the property that all central elements of A act on it as scalars. We denote the resultant coloured ribbon graphs by $\phi_j^{(k)}$ and $\zeta^{(k)}$ respectively. It is straightforward to obtain, for $k = 1$,

$$\begin{aligned} F(\phi_j^{(1)}) &= \chi_j(v^{-1})C_j : V_{i_1} \rightarrow V_{i_1}, \\ F(\zeta^{(1)}) &= v : V_{i_1} \rightarrow V_{i_1}, \end{aligned}$$

where C_j is the central element of A defined by equation (7) with the representation afforded by the A module V_j . The $v \in A$ is defined by (6), and $\chi_j(v^{-1})$ is its eigenvalue in V_j . Either by using induction on k or using the properties of coloured ribbon graphs discussed in *Remarks 6.4.2* of [11], we can show that

Lemma 1

$$\begin{aligned} F(\phi_j^{(k)}) &= \chi_j(v^{-1})\Delta^{(k-1)}(C_j) : V_{i_1} \otimes V_{i_2} \otimes \dots \otimes V_{i_k} \rightarrow V_{i_1} \otimes V_{i_2} \otimes \dots \otimes V_{i_k}, \\ F(\zeta^{(k)}) &= \Delta^{(k-1)}(v) : V_{i_1} \otimes V_{i_2} \otimes \dots \otimes V_{i_k} \rightarrow V_{i_1} \otimes V_{i_2} \otimes \dots \otimes V_{i_k}. \end{aligned} \quad (12)$$

These equations will play a central role in the construction of three - manifold invariants in the next section.

4 Three - manifold invariants

The interrelationship between knot theory and the theory of three - manifolds has long been known. In the early 1960s, Lickorish and Wallace proved that each framed link in S^3 determines a compact, closed, oriented 3 - manifold, and every such 3 - manifold is obtainable by surgery along a framed link in S^3 [15]. A framed link is a link with each of its component associated with an integer. We will work with the so called blackboard framing throughout this paper.

However, different framed links may yield homeomorphic 3 - manifolds upon surgery. A theorem due to Kirby and Crags[16] and refined by Fenn and Rourke in [17] states that orientation preserving homeomorphism classes of compact, closed,

oriented 3 - manifolds correspond bijectively to equivalence classes of framed links in S^3 , where the equivalence relation is generated by the Kirby moves given in Figure 3.

Figure 3.

It is also known that the special Kirby (\pm) - moves $\kappa_{\pm}^{(0)}$ plus either the Kirby $(+)$ - move κ_+ or the $(-)$ - move κ_- generate the entire Kirby calculus. We wish to point out that the Kirby moves are local operations, namely, each operation only alters a part of a given framed link while leaving the rest of the link unchanged.

Let L be a framed link (in the blackboard framing) embedded in S^3 , which consists of m components L_i , $i = 1, 2, \dots, m$. Surgery along L gives rise a three - manifold, which we denote by M_L . The basic idea of the Reshetikhin - Turaev construction of topological invariants of M_L using quantum groups is to make appropriately weighted sums of the framed link invariants of L associated with different representations in such a way that the final combinations are invariant under the Kirby moves.

The framed link L yields a unique ribbon graph by extending each of its component L_i to an annulus, which has L_i itself and an L'_i as its edges, where L'_i is a parallel copy of L_i such that the link number between the two is equal to the framing number of the latter. We denote this ribbon graph by Γ .

Let $(A, R, K_{2\rho})$ be a ribbon Hopf superalgebra, and $\mathcal{V}(\Lambda) = \{V(\lambda) | \lambda \in \Lambda\}$ a finite set consisting of such objects of Rep_A that all central elements of A act as scalars in each $V(\lambda)$, and $SD_q(V(\lambda)) \neq 0$. We also require that for each $\lambda \in \Lambda$, there exists a $\lambda^* \in \Lambda$ such that $V(\lambda^*)$ is isomorphic to the dual A module $V^*(\lambda)$ of $V(\lambda)$, and two modules $V(\lambda)$ and $V(\lambda')$ are isomorphic if and only if $\lambda = \lambda'$.

Colour the component of Γ associated with each L_i by the A module $V(\lambda^{(i)})$, and let $c = \{\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}\}$, where some $\lambda^{(j)}$'s may be equal. We denote the resultant coloured ribbon graph by $\Gamma(L, c)$. Let $\mathcal{C}(L, \Lambda)$ be the set of all distinct c 's. We define

$$\Sigma(L) = \sum_{c \in \mathcal{C}(L, \Lambda)} \Pi_{i=1}^m d_{\lambda^{(i)}} F(\Gamma(L, c)), \quad (13)$$

where $d_{\lambda^{(i)}}$ are a set of constants satisfying the following conditions:

1. $d_{\lambda} = d_{\lambda^*}$, $\forall \lambda \in \Lambda$;
2. Let C_{λ} be the central element of A defined by (7) with $V = V(\lambda)$, $\lambda \in \Lambda$. Define

$$\delta = v - \sum_{\lambda \in \Lambda} d_{\lambda} \chi_{\lambda}(v^{-1}) C_{\lambda}. \quad (14)$$

Then δ takes zero eigenvalues in all finite dimensional irreducible A modules with nonvanishing q - superdimensions.

$\Sigma(L)$ has the following important properties:

Theorem 2 $\Sigma(L)$ is independent of the orientation chosen for L , and also invariant under the positive Kirby moves κ_+ and $\kappa_+^{(0)}$.

In order to prove the theorem, we need the following

Lemma 2 *For any finite dimensional A - module W , and a linear A homomorphism $f : W \rightarrow W$, we have*

$$\text{Str}_W(K_{2\rho}\delta \circ f) = 0, \quad (15)$$

where δ is as defined by (14), and Str_W represents the supertrace taken over W .

Proof. The A - module W can always be decomposed into a direct sum of indecomposable submodules. A direct summand may contribute to the supertrace only if its image under f is contained in itself. Also the kernel of f does not contribute to the supertrace, thus we may assume that W is indecomposable, and the A homomorphism $f : W \rightarrow W$ is bijective.

If W is irreducible, (15) does not need any proof. Assume W is not irreducible but admits the composition series

$$W \supset W_1 \supset \{0\},$$

then we necessarily have $f(W_1) = W_1$, as otherwise W would not be indecomposable. We can now rewrite the supertrace $\text{Str}_W(K_{2\rho}\delta \circ f)$ as the sum $\text{Str}_{W/W_1}(K_{2\rho}\delta \circ f') + \text{Str}_{W_1}(K_{2\rho}\delta \circ f|_{W_1})$, where $f' : W/W_1 \rightarrow W/W_1$ is the map naturally induced by $f : W \rightarrow W$, and $f|_{W_1} : W_1 \rightarrow W_1$ is the restriction of f to W_1 , which again is bijective. Since both W/W_1 and W_1 are irreducible, the supertraces in the sum vanish separately.

To proceed further, we use induction on the length of the composition series for W . For a given composition series

$$W \supset W_1 \supset W_2 \supset \dots \supset W_t \supset \{0\},$$

we have another equivalent one

$$W \supset X_1 \supset X_2 \supset \dots \supset X_t \supset \{0\},$$

where X_i is the image of W_i under f . $W_1 \cap X_1$ is again a submodule of W , and its image under f is identical to itself. Let us rewrite the supertrace $\text{Str}_W(K_{2\rho}\delta \circ f)$ as

$$\text{Str}_W(K_{2\rho}\delta \circ f) = \text{Str}_{W/(W_1 \cap X_1)}(K_{2\rho}\delta \circ \tilde{f}) + \text{Str}_{W_1 \cap X_1}(K_{2\rho}\delta \circ f|_{W_1 \cap X_1}), \quad (16)$$

where $\tilde{f} : W/(W_1 \cap X_1) \rightarrow W/(W_1 \cap X_1)$ is the map induced by f , and $f|_{W_1 \cap X_1} : W_1 \cap X_1 \rightarrow W_1 \cap X_1$ is the restriction of f to $W_1 \cap X_1$. The second term on the right hand side of (16) vanishes following the induction hypothesis. To consider the first term, we need to examine the two cases $W_1 = X_1$ and $W_1 \neq X_1$ separately. In the former case, $W/(W_1 \cap X_1) = W/W_1$ is irreducible, hence $\text{Str}_{W/(W_1 \cap X_1)}(K_{2\rho}\delta \circ \tilde{f}) = 0$. In the latter case, $W_1 \cap X_1$ is a maximal submodule of both W_1 and X_1 , and $W = W_1 + X_1$. Also, for any composition series for $W_1 \cap X_1$,

$$(W_1 \cap X_1) \supset Y_3 \supset \dots \supset Y_t \supset 0,$$

the following

$$W \supset W_1 \supset (W_1 \cap X_1) \supset Y_3 \supset \dots \supset Y_t \supset 0,$$

is another composition series for W . Now $W/(W_1 \cap X_1) = V^1 \oplus V^2$, where $V^1 = W_1/(W_1 \cap X_1)$ and $V^2 = X_1/(W_1 \cap X_1)$. Clearly, $\tilde{f}(V^1) = V^2$ and $\tilde{f}(V^2) = V^1$. Hence, we again have $\text{Str}_{W/(W_1 \cap X_1)}(K_{2\rho} \delta \circ \tilde{f}) = 0$, and this completes the proof of the lemma.

Now we turn to the proof of Theorem 2. The first statement of the theorem is easy to see: Reversing the orientation of any component L_i of L is equivalent to replacing the A module $V(\lambda^{(i)})$ colouring L_i by its dual $V(\lambda^{(i)*})$. As $d_{\lambda^{(i)}} = d_{\lambda^{(i)*}}$, $\Sigma(L)$ is not affected.

To prove the second part of Theorem 2, we consider the framed links L and L' given in Figure 4,

Figure 4.

which are related to each other by a Kirby (+) - move κ_+ . The T appearing in both L and L' is an arbitrary oriented (k, k) tangle. It gives rise to a ribbon graph, which we colour by c such that the A modules $V(\lambda^{(1)})$, $V(\lambda^{(2)})$, ..., $V(\lambda^{(k)})$ are respectively assigned to the k open strands both on the top and down the bottom. We denote this coloured ribbon graph by $\Gamma(T; c)$. The colour of $\Gamma(T; c)$ induces a unique colour for the ribbon graph associated with L' . To colour the ribbon graph arising from L , we also need to assign $V(\mu)$, $\mu \in \Lambda$ to the annulus corresponding to the framing +1 unknot. Now

$$\begin{aligned} F(\Gamma(L; c, \mu)) &= \text{Str}_{V(\lambda^{(1)}) \otimes \dots \otimes V(\lambda^{(k)})} \left\{ K_{2\rho} F(\phi_\mu^{(k)}) \circ F(\Gamma(T; c)) \right\}, \\ F(\Gamma(L'; c)) &= \text{Str}_{V(\lambda^{(1)}) \otimes \dots \otimes V(\lambda^{(k)})} \left\{ K_{2\rho} F(\zeta^{(k)}) \circ F(\Gamma(T; c)) \right\}, \end{aligned}$$

where we have used equation (12) and the fact that $F(\phi_\mu^{(k)})$ and $F(\zeta^{(k)})$ both commute with $F(\Gamma(T; c))$. Inserting these equations in (13) we arrive at

$$\begin{aligned} \Sigma(L) &= \sum_c \prod_{i=1}^k d_{\lambda^{(i)}} \text{Str}_{V(\lambda^{(1)}) \otimes \dots \otimes V(\lambda^{(k)})} \left\{ K_{2\rho} \sum_\mu d_\mu \chi_\mu(v^{-1}) C_\mu \circ F(\Gamma(T; c)) \right\}, \\ \Sigma(L') &= \sum_c \prod_{i=1}^k d_{\lambda^{(i)}} \text{Str}_{V(\lambda^{(1)}) \otimes \dots \otimes V(\lambda^{(k)})} \left\{ K_{2\rho} v \circ F(\Gamma(T; c)) \right\}. \end{aligned}$$

Hence

$$\begin{aligned} \Sigma(L') - \Sigma(L) &= \sum_c \prod_{i=1}^k d_{\lambda^{(i)}} \text{Str}_{V(\lambda^{(1)}) \otimes \dots \otimes V(\lambda^{(k)})} \left\{ K_{2\rho} \delta \circ F(\Gamma(T; c)) \right\} \\ &= 0, \end{aligned} \tag{17}$$

where the last equality follows from Lemma 2. The invariance of Σ under the special Kirby (+) - move $\kappa_+^{(0)}$ is a direct consequence of the defining property of δ .

Let L be an oriented framed link of m components, and L' be the framed link obtained by applying once the special Kirby (−) move, namely, adding a framing -1 unknot to L . We consider the matrix $A_L = (a_{ij})_{m \times m}$ defined in the following way: a_{ii}

equals the framing number of the i -th component of L , and a_{ij} is equal to the linking number between the i -th and j -th component of L if $i \neq j$. If M_L is the three - manifold obtained by surgery along L , and the W_L is the four manifold bounded by M_L , then the matrix A_L can be interpreted as the intersection form on the second homology group $H_2(W_L, \mathbf{Z})$. Let $\sigma(A_L)$ be the number of nonpositive eigenvalues of A_L , then it is clear that $\sigma(A_L) = \sigma(A_{L'}) - 1$, while the positive Kirby moves leave $\sigma(A_L)$ unchanged.

Let

$$z = \sum_{\lambda \in \Lambda} d_\lambda \chi_\lambda(v) SD_q(\lambda). \quad (18)$$

If $z \neq 0$, we define

$$\mathcal{F}(M_L) = z^{-\sigma(A_L)} \Sigma(L). \quad (19)$$

Note that

$$\Sigma(L') = z \Sigma(L),$$

that is, under a special Kirby $(-)$ move, Σ is scaled by z . From the properties of $\sigma(A_L)$ discussed above we conclude that $\mathcal{F}(M_L)$ is invariant under the special Kirby (\pm) moves $\kappa_\pm^{(0)}$ and the Kirby $(+)$ move κ_+ . Since they generate the entire Kirby calculus, $\mathcal{F}(M_L)$ defines a topological invariant of the three - manifold M_L .

Before closing this section, we remark that any nongraded Hopf algebra B may be considered as a \mathbf{Z}_2 graded Hopf algebra A with $A_0 = B$, and $A_1 = 0$. Therefore the construction for three - manifold invariants developed in this section works equally well in the nongraded case. In fact one of the examples which we will study in the next section is the nongraded quantum group $U_q(gl(2))$ at odd roots of unity.

The conditions imposed on the d_λ 's are sufficient to ensure that Σ is invariant under the Kirby moves κ_+ and $\kappa_+^{(0)}$, as shown by Theorem 2. To implement these conditions, it is only necessary to study the irreducible A - modules with nonvanishing q - superdimensions, or more precisely, the eigenvalues of the central elements C_λ in these modules. This of course is still a rather hard problem, as in typical cases, A is a quantum supergroup at a root of unity, and its irreps are not well understood[31][32] in general. However, for some quantum (super)groups like $U_q(gl(2|1))$ etc., the irreps have been completely classified. In these case, our construction is easier to apply than that of [18], which require, besides other things, the classification of indecomposable representations of A .

5 Examples

In this section we apply the general method developed in the last section to the quantum supergroup $U_q(gl(2|1))$ at odd roots of unity to explicitly construct topological invariants of three - manifolds. But for the purpose of illustrating how the method works, we first consider $U_q(gl(2))$.

5.1 Invariants arising from $U_q(gl(2))$ at odd roots of unity

The three - manifold invariants associated with $U_q(sl(2))$ at even roots of unity were constructed by Reshetikhin and Turaev in Ref. [18], and they turned out to be equivalent to Witten's invariants arising from $su(2)$ Chern - Simons quantum field theory. These invariants were also explicitly computed for three - manifolds like the Lens spaces etc. using both the Chern - Simons theory approach[19] – [22] and the algebraic approach[33]. Therefore one has a reasonably good understanding of these invariants. Following a similar method as that of Ref. [15], three - manifold invariants were also constructed at odd roots of unity in [34]. Below is an alternative derivation of these invariants. As we will see, some new features also arise from this exercise.

Let us assume that

$$q = \exp(i2\pi/N), \quad N = 2r + 1, \quad r \in \mathbf{Z}_+. \quad (20)$$

The underlying algebra of $U_q(gl(2))$ is generated by $\{e, f, t_1^{\pm 1}, t_2^{\pm 1}\}$ subject to the constraints

$$\begin{aligned} t_i t_j &= t_j t_i, & t_i t_i^{-1} &= 1, \\ t_i e t_i^{-1} &= q^{\delta_i, 1 - \delta_i, 2} e, & t_i f t_i^{-1} &= q^{-\delta_i, 1 + \delta_i, 2} f, \\ [e, f] &= (k - k^{-1})/(q - q^{-1}), & k &= t_1 t_2^{-1}. \end{aligned} \quad (21)$$

We will also impose the extra relations

$$(t_i)^N = 1, \quad (e)^N = (f)^N = 0. \quad (22)$$

The algebra has the structures of a ribbon Hopf algebra with the co - multiplication

$$\begin{aligned} \Delta(e) &= e \otimes k + 1 \otimes e, \\ \Delta(f) &= f \otimes 1 + k^{-1} \otimes f \\ \Delta(t_i^{\pm 1}) &= t_i^{\pm 1} \otimes t_i^{\pm 1}; \end{aligned}$$

the co - unit

$$\epsilon(e) = \epsilon(f) = 0, \quad \epsilon(t_i^{\pm 1}) = \epsilon(1) = 1;$$

and the antipode

$$S(e) = -ek^{-1}, \quad S(f) = -kf, \quad S(t_i^{\pm 1}) = t_i^{\mp 1}.$$

The universal R matrix of $U_q(gl(2))$ reads

$$R = \sum_{\theta, \sigma \in \mathbf{Z}_N} t_1^\theta t_2^\sigma \otimes P_1[\theta] P_2[\sigma] \sum_{\mu \in \mathbf{Z}_N} \frac{[(q - q^{-1})e \otimes f]^\mu}{[\mu]_q!}, \quad (23)$$

where

$$\begin{aligned} P_i[\theta] &= \Pi_{\theta \neq \sigma \in \mathbf{Z}_N} \frac{t_i - q^\sigma}{q^\theta - q^\sigma}, \\ [\nu]_q! &= \begin{cases} \prod_{i=1}^\nu [(1 - q^{-2i})/(1 - q^{-2})], & \nu > 0, \\ 1, & \nu = 0; \end{cases} \end{aligned}$$

and

$$K_{2\rho} = k.$$

Each irreducible $U_q(gl(2))$ - module $V(\lambda)$ is uniquely characterized by a highest weight $\lambda = (\lambda_1, \lambda_2)$, $\lambda_1, \lambda_2 \in \mathbf{Z}_N$. A basis for $V(\lambda)$ is given by $\{v_i^\lambda | i = 0, 1, \dots, n_\lambda\}$, where $n_\lambda \in \mathbf{Z}_N$ is defined by $n_\lambda \equiv \lambda_1 - \lambda_2 \pmod{N}$. The actions of the $U_q(gl(2))$ generators on $V(\lambda)$ is defined by

$$\begin{aligned} ev_0^\lambda &= 0, & t_i v_0^\lambda &= q^{\lambda_i} v_0^\lambda, \\ fv_i^\lambda &= v_{i+1}^\lambda, & fv_{n_\lambda}^\lambda &= 0. \end{aligned}$$

The q - dimension of $V(\lambda)$ is given by

$$D_q(\lambda) = \frac{q^{n_\lambda+1} - q^{-n_\lambda-1}}{q - q^{-1}},$$

which vanishes when $n_\lambda = N - 1$. Therefore, for the purpose of constructing three - manifold invariants, we only need to consider the irreps with highest weights in $\Lambda = \{\lambda | n_\lambda \neq N - 1\}$.

Let $V(\lambda^*)$ be the dual $U_q(gl(2))$ module of $V(\lambda)$. It is easy to see that $\lambda^* = (N - \lambda_2, N - \lambda_1)$. Applying δ to $V(\lambda^*)$, and requiring that its eigenvalue vanish, we arrive at the following equations

$$q^{-\lambda_1(\lambda_1+1)-\lambda_2(\lambda_2-1)} = \sum_{\mu \in \Lambda} d_\mu q^{\mu_1(\mu_1-2\lambda_1)+\mu_2(\mu_2-2\lambda_2)} \sum_{\nu=0}^{n_\mu} q^{2\nu(n_\lambda+1)}, \quad \lambda \in \Lambda.$$

Simultaneously replacing λ by λ^* and μ by μ^* leaves the form of the equations intact, but changing d_μ to d_{μ^*} . However, since the equations do not determine the d_μ 's uniquely, the condition $d_\mu = d_{\mu^*}$ still needs to be imposed on the solutions.

Solving the equations under the condition $d_\mu = d_{\mu^*}$, we obtain

$$d_\lambda = q^{\frac{N^2+1}{2}} \left(\frac{G_{N-1}}{N} \right)^2 q^{-\lambda_1(\lambda_1+1)-\lambda_2(\lambda_2-1)} \left[q^{\lambda_1^2+\lambda_2^2} + x_{n_\lambda+1} + x_{N-n_\lambda-1} \right],$$

where the x 's are arbitrary complex numbers, and G_{N-1} is the $k = N - 1$ case of the Gauss sum defined by

$$G_k = \sum_{\nu=0}^{N-1} q^{k\nu^2}, \quad k \in \mathbf{Z}_N.$$

The corresponding z can now be readily worked out, and we have

$$z = -q^3 \left(\frac{G_{N-1}}{\sqrt{N}} \right)^4,$$

which is independent of the x 's. Inserting the d 's and z in equation (19), we arrive at the following three - manifold invariant

$$\mathcal{F}(M_L) = \left\{ -q^3 \left(\frac{G_{N-1}}{\sqrt{N}} \right)^4 \right\}^{-\sigma(A_L)} \sum_{c \in \mathcal{C}(L, \Lambda)} \prod_{i=1}^m d_{\lambda^{(i)}} F(\Gamma(L, c)).$$

It is in general a very difficult problem to explicitly compute this invariant for given manifolds. As a matter of fact, even the widely studied Witten - Reshetikhin - Turaev invariant has only been computed for some very simple manifolds like the Lens spaces, Seifert manifolds etc.. We hope, in the future, to carry out some explicit computations on this invariant and the one derived from $U_q(gl(2|1))$ in the next subsection. Here we merely present a simple example. For the three - manifold obtained by surgery along the framed knot

Figure 5.

with k crossings and p curls, where $k, p \in \mathbf{Z}$, and k is odd, we have

$$\begin{aligned} \mathcal{F} = & q^{(N+1)^2/2} G_{2k+2p} \left(\frac{G_{N-1}}{N} \right)^2 \left\{ -q^3 \left(\frac{G_{N-1}}{\sqrt{N}} \right)^4 \right\}^{-\sigma(k+p)} \\ & \times \sum_{\mu=0}^{N-1} q^{(k+p)\mu[1-\mu(N^2-1)/2]-k\mu(\mu+2)-\mu} \sum_{\nu=0}^{\mu} (-1)^{\mu+\nu+1} q^{k\nu(\nu+1)} \frac{q^{2\nu+1} - q^{-2\nu-1}}{q - q^{-1}}. \end{aligned}$$

Note that the free parameters x_μ miraculously cancel out among themselves in this particular example. However, we do not know whether this is true or not in general. In any case, the existence of these free parameters is a great advantage: Appropriately choosing the x 's will render some of the d 's vanishing. Then the corresponding irreps will not play any role in \mathcal{F} , and can be ignored. This can greatly simplify the computation of \mathcal{F} .

5.2 Invariants arising from $U_q(gl(2|1))$

5.2.1 $U_q^{(r)}(gl(2|1))$

As a \mathbf{Z}_2 graded algebra over the complex field \mathbf{C} , the quantum supergroup $U_q(gl(2|1))$ is generated by $\{e_i, f_i, t_a, t_a^{-1} \mid i = 1, 2, a = 1, 2, 3\}$ with the following relations

$$\begin{aligned} t_a t_b &= t_b t_a, & t_a t_a^{-1} &= 1, & a, b &= 1, 2, 3, \\ k_1 &= t_1 t_2^{-1}, & k_2 &= t_2 t_3, & k_3 &= t_1 t_2 t_3, \\ k_i e_j k_i^{-1} &= q^{a_{ij}} e_j, & k_i f_j k_i^{-1} &= q^{-a_{ij}} f_j, & [e_i, f_j] &= \frac{\delta_{ij}(k_i - k_i^{-1})}{q - q^{-1}}, \\ k_3 e_i k_3^{-1} &= e_i, & k_3 f_i k_3^{-1} &= f_i, & i, j &= 1, 2; \end{aligned} \tag{24}$$

$$\begin{aligned} (e_2)^2 &= (f_2)^2 = 0, \\ (e_1)^2 e_2 - ((q - q^{-1}) e_1 e_2 e_1 + e_2 (e_1)^2) &= 0, \\ (f_1)^2 f_2 - (q - q^{-1}) f_1 f_2 f_1 + f_2 (f_1)^2 &= 0, \end{aligned} \tag{25}$$

where q is a nonvanishing complex number, and the matrix (a_{ij}) is given by $\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$.

The $[,]$ is the standard graded brackets, and the gradation is defined by $[t_a] = [e_1] = [f_1] = 0$, $[e_2] = [f_2] = 1$.

$U_q(gl(2|1))$ has the structures of a \mathbf{Z}_2 graded Hopf algebra, with the co-multiplication $\Delta : U_q(gl(2|1)) \rightarrow U_q(gl(2|1)) \otimes U_q(gl(2|1))$,

$$\begin{aligned} \Delta(e_i) &= e_i \otimes k_i + 1 \otimes e_i, \\ \Delta(f_i) &= f_i \otimes 1 + k_i^{-1} \otimes f_i, \\ \Delta(t_a^{\pm 1}) &= t_a^{\pm 1} \otimes t_a^{\pm 1}; \end{aligned}$$

the co - unit $\epsilon : U_q(gl(2|1)) \rightarrow \mathbf{C}$,

$$\begin{aligned} \epsilon(e_i) &= \epsilon(f_i) = 0, \\ \epsilon(t_a^{\pm 1}) &= \epsilon(1) = 1; \end{aligned}$$

and the antipode $S : U_q(gl(2|1)) \rightarrow U_q(gl(2|1))$,

$$\begin{aligned} S(e_i) &= -e_i k_i^{-1}, \\ S(f_i) &= -k_i f_i, \\ S(t_a^{\pm 1}) &= t_a^{\mp 1}. \end{aligned}$$

In order to construct three - manifold invariants, we require q be a root of unity. For the sake of simplicity we again assume that q is given by (20). Note that although $U_q(gl(2|1))$ is a ribbon Hopf superalgebra at generic q [29], it does not admit a universal R matrix in the present case.

To circumvent this problem, we observe that the following elements $\{(e_1)^N, (f_1)^N, (t_a)^{\pm N} - 1, a = 1, 2, 3\}$ generate a double sided Hopf ideal \mathcal{J} of $U_q(gl(2|1))$, thus the quotient $U_q(gl(2|1))/\mathcal{J}$ is still a \mathbf{Z}_2 graded Hopf algebra, which we denote by $U_q^{(r)}(gl(2|1))$. The underlying algebra of $U_q^{(r)}(gl(2|1))$ may be regarded as defined by (24), (25) together with the following relations

$$(e_1)^N = (f_1)^N = 0, \quad (t_a)^{\pm N} = 1, \quad (26)$$

while the co - multiplication Δ , co - unit ϵ and the antipode S remain the same.

Now $U_q^{(r)}(gl(2|1))$ has the structures of a ribbon Hopf superalgebra, with

$$\begin{aligned} K_{2\rho} &= k_2^{-2}, \\ R &= \sum_{\nu, \theta, \sigma \in \mathbf{Z}_N} t_1^\nu t_2^\theta t_3^{-\sigma} \otimes P_1[\nu] P_2[\theta] P_3[\sigma] \tilde{R}. \end{aligned} \quad (27)$$

In the above equation

$$\begin{aligned} P_a[\theta] &= \prod_{\theta \neq \sigma \in \mathbf{Z}_N} \frac{t_a - q^\sigma}{q^\theta - q^\sigma}, \\ \tilde{R} &= \sum_{\theta \in \mathbf{Z}_N} \frac{[(q - q^{-1})e_1 \otimes f_1]^\theta}{[\theta]_q!} [1 \otimes 1 - (q - q^{-1})E \otimes F] [1 \otimes 1 - (q - q^{-1})e_2 \otimes f_2], \end{aligned}$$

where

$$\begin{aligned} E &= e_1 e_2 - q^{-1} e_2 e_1, \\ F &= f_2 f_1 - q f_1 f_2. \end{aligned}$$

5.2.2 Irreducible representations

In Ref.[30], we explicitly constructed all the irreducible representations of $U_q(gl(2|1))$ using the so - called induced module construction. From the results of that publication, we can easily extract all the information about $U_q^{(r)}(gl(2|1))$ irreps, some main features of which are listed below:

1. $U_q^{(r)}(gl(2|1))$ admits a finite number of irreps, and every irrep is finite dimensional;
2. Every irrep is of highest weight type, namely, there exist a unique highest weight vector and a unique lowest weight vector;
3. The t_a 's can be diagonalized simultaneously in every irrep.

To construct an irrep of $U_q^{(r)}(gl(2|1))$, we start with a one dimensional module $\{v^\lambda\}$ of the Borel subalgebra generated by $\{e_i, t_a\}$ such that

$$\begin{aligned} e_i v^\lambda &= 0, \quad i = 1, 2, \\ k_a v^\lambda &= q^{\lambda_a} v^\lambda, \quad a = 1, 2, 3, \end{aligned} \quad (28)$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, and we require that

$$\lambda_a \in \mathbf{Z}_N, \quad a = 1, 2, 3.$$

Let us now construct an irreducible module $V_0(\lambda)$ over the even quantum subgroup $U_q(gl(2) \oplus u(1)) \subset U_q^{(r)}(gl(2|1))$ generated by $\{e_1, f_1, t_a^{\pm 1}\}$:

$$V_0(\lambda) = \{(f_1)^i v^\lambda \mid i = 0, 1, \dots, \lambda_1\}.$$

Then we build the following $U_q^{(r)}(gl(2|1))$ module from $V_0(\lambda)$

$$\bar{V}(\lambda) = V_0(\lambda) \oplus f_2 V_0(\lambda) \oplus F V_0(\lambda) \oplus F f_2 V_0(\lambda).$$

Note that $\bar{V}(\lambda)$ is not irreducible in general, but it is indecomposable, and is generated by a single vector v^λ . Therefore all the central elements of $U_q^{(r)}(gl(2|1))$ act as scalars in $\bar{V}(\lambda)$. Let $M(\lambda)$ be the maximal proper submodule of $\bar{V}(\lambda)$. Define

$$V(\lambda) = \bar{V}(\lambda)/M(\lambda).$$

Then $V(\lambda)$ furnishes an irreducible $U_q^{(r)}(gl(2|1))$ module. Let

$$Q(\lambda) = \frac{q^{\lambda_2} - q^{-\lambda_2}}{q - q^{-1}} \frac{q^{\lambda_1 + \lambda_2 + 1} - q^{-\lambda_1 - \lambda_2 - 1}}{q - q^{-1}}.$$

We call λ typical if $Q(\lambda) \neq 0$, and atypical otherwise. Now we give a complete classification of all the irreps.

1. For λ typical:

$$\begin{aligned} V(\lambda) &= \bar{V}(\lambda); \\ SD_q(\lambda) &= 0, \\ \chi_\lambda(v) &= q^{-\lambda_3^2 - 2\lambda_2(\lambda_1 + \lambda_2 + 1)}. \end{aligned}$$

2. $\lambda_2 = 0$; v^λ even:

$$\begin{aligned} V(\lambda) &= V_0(\lambda) \oplus \{f_2(f_1)^i v^\lambda | i = 1, 2, \dots, \lambda_1\}, \\ SD_q(\lambda) &= 1, \\ \chi_\lambda(v) &= q^{-\lambda_3^2}. \end{aligned}$$

3. $\lambda_2 \neq 0$, $\lambda_1 + \lambda_2 + 1 \equiv 0 \pmod{N}$; v^λ odd:

$$\begin{aligned} V(\lambda) &= V_0(\lambda) \oplus f_2 V_0(\lambda) \oplus F v^\lambda, \\ SD_q(\lambda) &= 1, \\ \chi_\lambda(v) &= q^{-\lambda_3^2}. \end{aligned}$$

It is useful to observe that the dual of a typical irrep is also typical, and the irreps of dimension greater than one in the second group are dual to irreps in the third group.

5.2.3 Three - manifold invariants arising from $U_q^{(r)}(gl(2|1))$

Let Λ be the set of all atypical λ 's, namely, $\Lambda = \{\lambda = (\lambda_1, \lambda_2, \lambda_3) | \lambda_a \in \mathbf{Z}_N, Q(\lambda) = 0\}$. Let $a_{\theta, \omega}$, $\theta, \omega \in \mathbf{Z}_N$ be a set of complex numbers satisfying

$$\begin{aligned} a_{0, \omega} &= a_{0, N-\omega}, \\ \frac{1}{N} q^{\omega^2} G &= a_{0, \omega} + \sum_{\theta=1}^{N-1} (a_{\theta, \omega} + a_{N-\theta, N-\omega}). \end{aligned} \tag{29}$$

Define

$$\begin{aligned} d_\lambda &= q^{-\lambda_3^2} a_{\lambda_1, \lambda_3}, \quad \text{if } \lambda_2 = 0, \\ d_\lambda &= q^{-\lambda_3^2} a_{\lambda_2, N-\lambda_3}, \quad \text{if } \lambda_1 = N - \lambda_2 - 1, \quad \lambda_2 \neq 0. \end{aligned} \tag{30}$$

Then it can be shown that the central element

$$\delta = v - \sum_{\lambda \in \Lambda} d_\lambda \chi_\lambda(v^{-1}) C_\lambda,$$

takes zero eigenvalue in all atypical irreps of $U_q^{(r)}(gl(2|1))$. It is easy to work out the corresponding z as defined by (18), and we have

$$z = \left(\frac{G_{N-1}}{\sqrt{N}} \right)^2.$$

Applying (30) and z to equation (19), we obtain the following three - manifold invariant

$$\mathcal{F}(M_L) = \left(\frac{G_{N-1}}{\sqrt{N}} \right)^{-2\sigma(A_L)} \sum_{c \in \mathcal{C}(L, \Lambda)} \prod_{i=1}^m d_{\lambda^{(i)}} F(\Gamma(L, c)). \tag{31}$$

Explicit computations of this invariant for particular manifolds of interest is a problem meriting investigation on its own right, and we hope to return to it in the future.

Here we consider the special case with $a_{\mu,\nu} = 0, \forall \mu \neq 0$. Now only one - dimensional representations contribute to the invariant, and this fact makes \mathcal{F} very easy to compute. For any framed link L , we have

$$\mathcal{F}(M_L) = \frac{1}{N^{m/2}} \left(\frac{G_{N-1}}{\sqrt{N}} \right)^{m-2\sigma(A_L)} \sum_{\lambda_3^{(1)}, \dots, \lambda_3^{(m)}=0}^{N-1} q^{\langle \Lambda_3 | A_L | \Lambda_3 \rangle}, \quad (32)$$

where the matrix A_L is as defined in section 4, and

$$\langle \Lambda_3 | A_L | \Lambda_3 \rangle = \sum_{i,j=1}^m \lambda_3^{(i)} (A_L)_{ij} \lambda_3^{(j)}.$$

Dirct calculations can confirm that (32) is indeed invariant under all the Kirby moves. It is also worth observing that this invariant is closely related to the invariant discussed in [35], where the latter was also shown to be obtainable from $U(1)$ Chern - Simons theory.

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